Úkol č. 6

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$$\int_0^1 \frac{x^2 \log \frac{1}{x}}{1+x}\,dx = \int_0^1 \frac{x^2 \log \frac{1}{x}}{1+x}\,dx = \int_0^1 x^2 \log\frac{1}{x} \sum_{n=0}^{\infty} (-x)^n \,dx$$

geometrická řada $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n$

$$= \int_0^1 \sum_{n=0}^{\infty} (-1)^n x^{n+2} \log\frac{1}{x}\,dx = \sum_{n=0}^{\infty} \int_0^1 (-1)^n x^{n+2} \log\frac{1}{x}\,dx$$

$$\left| \sum_{n=0}^{N} (-1)^n x^{n+2} \log\frac{1}{x} \right| \leq \sum_{n=0}^{\infty} x^{n+2} \log\frac{1}{x} = x^2 \frac{\log\frac{1}{x}}{1-x} =: g(x)$$

Protože
- g je spojitá na intervalu $(0,1)$,
- $\lim_{x\to 0+} g(x) = 0$, $\lim_{x \to 1-} g(x) = 1$

platí, že g je omezená na $[0,1]$ a tedy

$\int |g| = \int g < \infty$. ($\Rightarrow g \in \mathcal{L}_1$)

Můžeme tedy prohodit $\sum$ a $\int$ pomocí Lebesgueovy věty

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \int_0^1 x^{n+2} \log x \, dx$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \cdot \left(\overbrace{\left[\frac{x^{n+3}}{n+1} \cdot \log x \right]_0^1}^{0-0} - \int_0^1 \frac{x^{n+3}}{n+3} \cdot \frac{1}{x} dx \right)$$

$$= \sum_{n=0}^{\infty} (-1)^{n} \left[\frac{x^{n+4}}{(n+3)^2} \right]_0^1 = \sum_{n=0}^{\infty} (-1)^{n} \frac{1}{(n+3)^2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} - 1 + \frac{1}{4} = \underline{\underline{\frac{\pi^2}{12} - \frac{3}{4}}}$$

$$\shortparallel$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{1}{2} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{12}$$